

A Note on Inextensible Flows of Curves on Oriented Surface

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Abstract

In this paper, the general formulation for inextensible flows of curves on oriented surface in \mathbb{R}^3 is investigated. The necessary and sufficient conditions for inextensible curve flow lying on an oriented surface are expressed as a partial differential equation involving the geodesic curvature and the geodesic torsion. Moreover, some special cases of inextensible curves on oriented surface are given.

Mathematics Subject Classification (2010): 53C44, 53A04, 53A05, 53A35.

Keywords: Curvature flows, inextensible, oriented surface.

1 Introduction

It is well known that many nonlinear phenomena in physics, chemistry and biology are described by dynamics of shapes, such as curves and surfaces. The evolution of curve and surface has significant applications in computer vision and image processing. The time evolution of a curve or surface generated by its corresponding flow in \mathbb{R}^3 -for this reason we shall also refer to curve and surface evolutions as flows throughout this article- is said to be inextensible if, in the former case, its arclength is preserved, and in the latter case, if its intrinsic curvature is preserved. Physically, the inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of the physical applications. For example, both Chirikjian and Burdick [1] and Mochiyama et al. [2] study the shape control of hyper-redundant, or snake-like robots.

The inextensible curve and surface flows also arise in the context of many problems in computer vision [3], [4], computer animation [5] and even structural mechanics [6]. There have been a lot of studies in the literature on plane curve flows, particularly on evolving curves in the direction of their curvature vector field (referred to by various names such as "curve shortening", flow by curvature" and "heat flow"). Particularly relevant to this paper are the methods developed by Gage and Hamilton [7] and Grayson [8] for studying the shrinking of closed plane curves to circle via heat equation.

The distinction between heat flows and inextensible flows of planar curves were elaborated in detail, and some examples of the latter were given by [9]. Also, a general formulation for inextensible flows of curves and developable surfaces in \mathbb{R}^3 are exposed by [10].

In this paper, we develop the general formulation for inextensible flows of curves according to Darboux frame in \mathbb{R}^3 . Necessary and sufficient conditions for an inextensible curve flow are expressed as a partial differential equation involving the geodesic curvature and geodesic torsion.

2 Preliminaries

Let S be an oriented surface in three-dimensional Euclidean space E^3 and $\alpha(s)$ be a curve lying on the surface S . Suppose that the curve $\alpha(s)$ is spatial then there exists the Frenet frame $\{\vec{T}, \vec{N}, \vec{B}\}$ at each points of the curve where \vec{T} is unit tangent vector, \vec{N} is principal normal vector and \vec{B} is binormal vector, respectively. The Frenet equation of the curve $\alpha(s)$ is given by

$$\begin{aligned}\vec{T}' &= \kappa \vec{N} \\ \vec{N}' &= -\kappa \vec{T} + \tau \vec{B} \\ \vec{B}' &= -\tau \vec{N}\end{aligned}$$

where κ and τ are curvature and torsion of the curve $\alpha(s)$, respectively. Since the curve $\alpha(s)$ lies on the surface S there exists another frame of the curve $\alpha(s)$ which is called Darboux frame and denoted by $\{\vec{T}, \vec{g}, \vec{n}\}$. In this frame \vec{T} is the unit tangent of the curve, \vec{n} is the unit normal of the surface S and \vec{g} is a unit vector given by $\vec{g} = \vec{n} \times \vec{T}$. Since the unit tangent \vec{T} is common element of both Frenet frame and Darboux frame, the vectors $\vec{N}, \vec{B}, \vec{g}$ and \vec{n} lie on the same plane. So that the relations between these frames can be given as follows

$$\begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

where φ is the angle between the vectors \vec{g} and \vec{N} . The derivative formulae of

the Darboux frame is

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\vec{g}} \\ \dot{\vec{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ k_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix}$$

where k_g , k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsions, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve. The relations between the geodesic curvature, normal curvature, geodesic torsion and κ , τ are given as follows, [11]

$$k_g = \kappa \cos \varphi, \quad k_n = \kappa \sin \varphi, \quad \tau_g = \tau + \frac{d\varphi}{ds}.$$

Furthermore, the geodesic curvature k_g and geodesic torsion τ_g of curve $\alpha(s)$ can be calculated as follows, [11]

$$k_g = \left\langle \frac{d\vec{\alpha}}{ds}, \frac{d^2\vec{\alpha}}{ds^2} \times \vec{n} \right\rangle$$

$$\tau_g = \left\langle \frac{d\vec{\alpha}}{ds}, \vec{n} \times \frac{d\vec{n}}{ds} \right\rangle.$$

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface S the following relationships are well-known, [11]

- i- $\alpha(s)$ is a geodesic curve if and only if $k_g = 0$,
- ii- $\alpha(s)$ is a asymptotic line if and only if $k_n = 0$,
- iii- $\alpha(s)$ is a principal line if and only if $\tau_g = 0$.

Through the every point of the surface a geodesic passes in every direction. A geodesic is uniquely determined by an initial point and tangent at that point. All straight lines on a surface are geodesics.

Along all curved geodesics the principal normal coincides with the surface normal. Along asymptotic lines osculating planes and tangent planes coincide, along geodesics they are normal. Through a point of a non-developable surface pass two asymptotic lines which can be real or imaginary.

3 Inextensible Flows of Curve Lying on Oriented Surface

Throughout this paper, we suppose that

$$\alpha : [0, l] \times [0, w) \rightarrow M \subset E^3$$

is a one parameter family of differentiable curves on orientable surface M in E^3 , where l is the arclength of the initial curve. Let u be the curve parameterization variable, $0 \leq u \leq l$. If the speed of curve α is denoted by $v = \left\| \frac{\partial \vec{\alpha}}{\partial u} \right\|$ then the arclength of α is

$$S(u) = \int_0^u \left\| \frac{\partial \vec{\alpha}}{\partial u} \right\| du = \int_0^u v du. \quad (3.1)$$

The operator $\frac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}. \quad (3.2)$$

Thus, the arclength is $ds = v du$.

Definition 3.1 Let M be an orientable surface and α be a differentiable curve on M in E^3 . Any flow of the curve α with respect to Darboux frame $\{\vec{T}, \vec{g}, \vec{n}\}$ can be expressed following form:

$$\frac{\partial \vec{\alpha}}{\partial t} = f_1 \vec{T} + f_2 \vec{g} + f_3 \vec{n}. \quad (3.3)$$

Here, f_1 , f_2 and f_3 are scalar speed of the curve α . Let the arclength variation be

$$S(u, t) = \int_0^u v du. \quad (3.4)$$

In the Euclidean space the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} S(u, t) = \int_0^u \frac{\partial v}{\partial t} du = 0, \quad u \in [0, 1]. \quad (3.5)$$

Definition 3.2 A curve evolution $\alpha(u, t)$ and its flow $\frac{\partial \vec{\alpha}}{\partial t}$ on the oriented surface M in E^3 are said to be inextensible if

$$\frac{\partial}{\partial t} \left\| \frac{\partial \vec{\alpha}}{\partial u} \right\| = 0.$$

Now, we research the necessary and sufficient condition for inelastic curve flow. For this reason, we need to the following Lemma.

Lemma 3.1 In E^3 , let M be an orientable surface and $\{\vec{T}, \vec{g}, \vec{n}\}$ be a Darboux frame of α on M . There exists following relation between the scalar speed functions f_1 , f_2 , f_3 and the normal curvature k_n , geodesic curvature k_g of α the curve

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v k_g - f_3 v k_n. \quad (3.6)$$

Proof. Since $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute and $v^2 = \left\langle \frac{\partial \vec{\alpha}}{\partial u}, \frac{\partial \vec{\alpha}}{\partial u} \right\rangle$, we have

$$\begin{aligned} 2v \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} \left\langle \frac{\partial \vec{\alpha}}{\partial u}, \frac{\partial \vec{\alpha}}{\partial u} \right\rangle \\ &= 2 \left\langle \frac{\partial \vec{\alpha}}{\partial u}, \frac{\partial}{\partial t} \left(f_1 \vec{T} + f_2 \vec{g} + f_3 \vec{n} \right) \right\rangle \\ &= 2v \left(\frac{\partial f_1}{\partial u} - f_2 v k_g - f_3 v k_n \right). \end{aligned}$$

Thus, we reach

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v k_g - f_3 v k_n.$$

If we take in the conditions of being geodesic and asymptotic of a curve and Lemma 3.1, we give the following ■

Corollary 3.1 *If the curve is a geodesic curve or asymptotic curve, then there is following equations*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_3 v k_n$$

or

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v k_g,$$

respectively.

Theorem 3.1 *Let $\{\vec{T}, \vec{g}, \vec{n}\}$ be Darboux frame of the curve α on M and $\frac{\partial \vec{\alpha}}{\partial t} = f_1 \vec{T} + f_2 \vec{g} + f_3 \vec{n}$ be a differentiable flow of α in \mathbb{R}^3 . Then the flow is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = f_2 k_g + f_3 k_n. \quad (3.7)$$

Proof. Suppose that the curve flow is inextensible. From equations (3.4) and (3.6) for $u \in [0, l]$, we see that

$$\frac{\partial}{\partial t} S(u, t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u \left(\frac{\partial f_1}{\partial u} - f_2 v k_g - f_3 v k_n \right) du = 0. \quad (3.8)$$

Thus, it can be seen that

$$\frac{\partial f_1}{\partial u} = f_2 v k_g + f_3 v k_n. \quad (3.9)$$

Considering the last equation and (3.2), we reach

$$\frac{\partial f_1}{\partial s} = f_2 k_g + f_3 k_n.$$

Conversely, following similar way as above, the proof is completed. ■
From Theorem 3.1, we have following corollary.

Corollary 3.2 *i- Let the curve α is a geodesic curve on M . Then the curve flow is inextensible if and only if $\frac{\partial f_1}{\partial s} = f_3 k_n$.*

ii- Let the curve α is a asymptotic line on M . Then the curve flow is inextensible if and only if $\frac{\partial f_1}{\partial s} = f_2 k_g$.

Now, we restrict ourselves to the arclength parameterized curves. That is, $v = 1$ and the local coordinate u corresponds to the curve arclength s . We require the following Lemma.

Lemma 3.2 *Let M be an orientable surface in E^3 and $\{\vec{T}, \vec{g}, \vec{n}\}$ be a Darboux frame of the curve α on M . Then, the differentiations of $\{\vec{T}, \vec{g}, \vec{n}\}$ with respect to t is*

$$\begin{aligned}\frac{\partial \vec{T}}{\partial t} &= \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \vec{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \vec{n} \\ \frac{\partial \vec{g}}{\partial t} &= -\left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \vec{T} + \psi \vec{n} \\ \frac{\partial \vec{n}}{\partial t} &= -\left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \vec{T} - \psi \vec{g}\end{aligned}$$

where $\psi = \left\langle \frac{\partial \vec{g}}{\partial t}, \vec{n} \right\rangle$.

Proof. Since $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ are commutative, it seen that

$$\begin{aligned}\frac{\partial \vec{T}}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\partial \vec{\alpha}}{\partial s} \right) = \frac{\partial}{\partial s} \left(\frac{\partial \vec{\alpha}}{\partial t} \right) = \frac{\partial}{\partial s} \left(f_1 \vec{T} + f_2 \vec{g} + f_3 \vec{n} \right) \\ &= \frac{\partial f_1}{\partial s} \vec{T} + f_1 \frac{\partial \vec{T}}{\partial s} + \frac{\partial f_2}{\partial s} \vec{g} + f_2 \frac{\partial \vec{g}}{\partial s} + \frac{\partial f_3}{\partial s} \vec{n} + f_3 \frac{\partial \vec{n}}{\partial s}.\end{aligned}$$

Substituting the equation (3.7) into the last equation and using Theorem 3.1, we have

$$\frac{\partial \vec{T}}{\partial t} = \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \vec{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \vec{n}.$$

Now, let us differentiate the Darboux frame with respect to t as follows;

$$\begin{aligned}0 &= \frac{\partial}{\partial t} \langle \vec{T}, \vec{g} \rangle = \left\langle \frac{\partial \vec{T}}{\partial t}, \vec{g} \right\rangle + \left\langle \vec{T}, \frac{\partial \vec{g}}{\partial t} \right\rangle \\ &= \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \left\langle \vec{T}, \frac{\partial \vec{g}}{\partial t} \right\rangle\end{aligned}\tag{3.10}$$

$$\begin{aligned}0 &= \frac{\partial}{\partial t} \langle \vec{T}, \vec{n} \rangle = \left\langle \frac{\partial \vec{T}}{\partial t}, \vec{n} \right\rangle + \left\langle \vec{T}, \frac{\partial \vec{n}}{\partial t} \right\rangle \\ &= \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \left\langle \vec{T}, \frac{\partial \vec{n}}{\partial t} \right\rangle\end{aligned}\tag{3.11}$$

From (3.10) and (3.11), we have obtain

$$\frac{\partial \vec{g}}{\partial t} = - \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g \right) \vec{T} + \psi \vec{n}$$

and

$$\frac{\partial \vec{n}}{\partial t} = - \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g \right) \vec{T} - \psi \vec{g}$$

respectively, where $\psi = \left\langle \frac{\partial \vec{g}}{\partial t}, \vec{n} \right\rangle$. ■

If we take into consideration last Lemma, we have following corollary.

Corollary 3.3 *Let M be an orientable surface in E^3 .*

i- If the curve α is a geodesic curve, then

$$\begin{aligned} \frac{\partial \vec{T}}{\partial t} &= \left(\frac{\partial f_2}{\partial s} - f_3 \tau_g \right) \vec{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g \right) \vec{n}, \\ \frac{\partial \vec{g}}{\partial t} &= - \left(\frac{\partial f_2}{\partial s} - f_3 \tau_g \right) \vec{T} + \psi \vec{n}, \\ \frac{\partial \vec{n}}{\partial t} &= - \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g \right) \vec{T} - \psi \vec{g}, \end{aligned}$$

where $\psi = \left\langle \frac{\partial \vec{g}}{\partial t}, \vec{n} \right\rangle$.

ii- If the curve α is a asymptotic line, then

$$\begin{aligned} \frac{\partial \vec{T}}{\partial t} &= \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g \right) \vec{g} + \left(\frac{\partial f_3}{\partial s} + f_2 \tau_g \right) \vec{n}, \\ \frac{\partial \vec{g}}{\partial t} &= - \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g \right) \vec{T} + \psi \vec{n}, \\ \frac{\partial \vec{n}}{\partial t} &= - \left(\frac{\partial f_3}{\partial s} + f_2 \tau_g \right) \vec{T} - \psi \vec{g}, \end{aligned}$$

where $\psi = \left\langle \frac{\partial \vec{g}}{\partial t}, \vec{n} \right\rangle$.

iii- If the curve is a curvature line, then

$$\begin{aligned} \frac{\partial \vec{T}}{\partial t} &= \left(f_1 k_g + \frac{\partial f_2}{\partial s} \right) \vec{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} \right) \vec{n}, \\ \frac{\partial \vec{g}}{\partial t} &= - \left(f_1 k_g + \frac{\partial f_2}{\partial s} \right) \vec{T} + \psi \vec{n}, \\ \frac{\partial \vec{n}}{\partial t} &= - \left(f_1 k_n + \frac{\partial f_3}{\partial s} \right) \vec{T} - \psi \vec{g}, \end{aligned}$$

where $\psi = \left\langle \frac{\partial \vec{g}}{\partial t}, \vec{n} \right\rangle$.

Theorem 3.2 *Suppose that the curve flow $\frac{\partial \vec{\alpha}}{\partial t} = f_1 \vec{T} + f_2 \vec{g} + f_3 \vec{n}$ is inextensible on the orientable surface on M . In this case, the following partial differential equation are held:*

$$\begin{aligned} \frac{\partial k_g}{\partial t} &= \frac{\partial f_1}{\partial s} k_g + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s} \tau_g - f_3 \frac{\partial \tau_g}{\partial s} - f_1 k_n \tau_g - \frac{\partial f_3}{\partial s} \tau_g - f_2 \tau_g^2, \\ \frac{\partial k_n}{\partial t} &= \frac{\partial f_1}{\partial s} k_n + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + \frac{\partial f_2}{\partial s} \tau_g + f_2 \frac{\partial \tau_g}{\partial s} + f_1 k_g \tau_g + \frac{\partial f_2}{\partial s} \tau_g - f_3 \tau_g^2, \\ \frac{\partial \tau_g}{\partial t} &= -f_1 k_g k_n - \frac{\partial f_2}{\partial s} k_n + f_3 k_n \tau_g + \frac{\partial \psi}{\partial s}, \\ \psi k_n &= \left(-f_1 k_n - \frac{\partial f_3}{\partial s} - f_2 \tau_g \right) \tau_g, \\ \psi k_g &= \left(-f_1 k_g - \frac{\partial f_3}{\partial s} + f_3 \tau_g \right) \tau_g. \end{aligned}$$

Proof. Since $\frac{\partial}{\partial s} \frac{\partial \vec{T}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \vec{T}}{\partial s}$ we get

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial \vec{T}}{\partial t} &= \frac{\partial}{\partial s} \left[\left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g \right) \vec{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g \right) \vec{n} \right] \\ &= \left(\frac{\partial f_1}{\partial s} k_g + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s} \tau_g - f_3 \frac{\partial \tau_g}{\partial s} \right) \vec{g} + \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g \right) \frac{\partial \vec{g}}{\partial s} \\ &\quad + \left(\frac{\partial f_1}{\partial s} k_n + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + \frac{\partial f_2}{\partial s} \tau_g + f_2 \frac{\partial \tau_g}{\partial s} \right) \vec{n} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g \right) \frac{\partial \vec{n}}{\partial s} \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial \vec{T}}{\partial t} &= \left(\frac{\partial f_1}{\partial s} k_g + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s} \tau_g - f_3 \frac{\partial \tau_g}{\partial s} \right) \vec{g} + \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g \right) \left(-k_g \vec{T} + \tau_g \vec{n} \right) \\ &\quad + \left(\frac{\partial f_1}{\partial s} k_n + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + \frac{\partial f_2}{\partial s} \tau_g + f_2 \frac{\partial \tau_g}{\partial s} \right) \vec{n} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g \right) \left(-k_g \vec{T} - \tau_g \vec{g} \right) \end{aligned}$$

while

$$\frac{\partial}{\partial t} \frac{\partial \vec{T}}{\partial s} = \frac{\partial}{\partial t} (k_g \vec{g} + k_n \vec{n}) = \frac{\partial k_g}{\partial t} \vec{g} + k_g \frac{\partial \vec{g}}{\partial t} + \frac{\partial k_n}{\partial t} \vec{n} + k_n \frac{\partial \vec{n}}{\partial t}.$$

Thus, from the both of above two equations, we reach

$$\frac{\partial k_g}{\partial t} = \frac{\partial f_1}{\partial s} k_g + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s} \tau_g - f_3 \frac{\partial \tau_g}{\partial s} - f_1 k_n \tau_g - \frac{\partial f_3}{\partial s} \tau_g - f_2 \tau_g^2 \quad (3.12)$$

and

$$\frac{\partial k_n}{\partial t} = \frac{\partial f_1}{\partial s} k_n + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + \frac{\partial f_2}{\partial s} \tau_g + f_2 \frac{\partial \tau_g}{\partial s} + f_1 k_g \tau_g + \frac{\partial f_2}{\partial s} \tau_g - f_3 \tau_g^2. \quad (3.13)$$

Noting that $\frac{\partial}{\partial s} \frac{\partial \vec{g}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \vec{g}}{\partial s}$, it is seen that

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial \vec{g}}{\partial t} &= \frac{\partial}{\partial s} \left[- \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g \right) \vec{T} + \psi \vec{n} \right] \\ &= - \left(\frac{\partial f_1}{\partial s} k_g + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s} \tau_g - f_3 \frac{\partial \tau_g}{\partial s} \right) \vec{T} - \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g \right) (k_g \vec{g} + k_n \vec{n}) \\ &\quad + \frac{\partial \psi}{\partial s} \vec{n} + \psi \left(-k_n \vec{T} - \tau_g \vec{g} \right) \end{aligned}$$

while

$$\frac{\partial}{\partial t} \frac{\partial \vec{g}}{\partial s} = \frac{\partial}{\partial t} \left(-k_g \vec{T} + \tau_g \vec{n} \right) = -\frac{\partial k_g}{\partial t} \vec{T} - k_g \frac{\partial \vec{T}}{\partial t} + \frac{\partial \tau_g}{\partial t} \vec{n} + \tau_g \frac{\partial \vec{n}}{\partial t}.$$

Thus, we obtain

$$\frac{\partial k_g}{\partial t} = \frac{\partial f_1}{\partial s} k_g + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s} \tau_g - f_3 \frac{\partial \tau_g}{\partial s} + \psi k_n \quad (3.14)$$

and

$$\frac{\partial \tau_g}{\partial t} = -f_1 k_g k_n - \frac{\partial f_2}{\partial s} k_n + f_3 k_n \tau_g + \frac{\partial \psi}{\partial s}. \quad (3.15)$$

From the equations (3.12) and (3.14), it is seen that

$$\begin{aligned} \psi k_n &= -f_1 k_n \tau_g - \frac{\partial f_3}{\partial s} \tau_g - f_2 \tau_g^2 \\ &= \left(-f_1 k_n - \frac{\partial f_3}{\partial s} - f_2 \tau_g \right) \tau_g. \end{aligned}$$

By same way as above and considering $\frac{\partial}{\partial s} \frac{\partial \vec{n}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \vec{n}}{\partial s}$, we reach

$$\frac{\partial k_n}{\partial t} = \frac{\partial f_1}{\partial s} k_n + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + \frac{\partial f_2}{\partial s} \tau_g + f_2 \frac{\partial \tau_g}{\partial s} - \psi k_g. \quad (3.16)$$

Hence, from the equations (3.13) and (3.16), we get

$$\psi k_g = \left(-f_1 k_g - \frac{\partial f_2}{\partial s} + f_3 \tau_g \right) \tau_g.$$

■

Thus, we give the following corollary from last theorem.

Corollary 3.4 *Let M be an orientable surface in E^3 .*

i- If the curve α is a geodesic curve on M , then we have

$$\begin{aligned} \frac{\partial k_n}{\partial t} &= \frac{\partial f_1}{\partial s} k_n + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + \frac{\partial f_2}{\partial s} \tau_g + f_2 \frac{\partial \tau_g}{\partial s} + \frac{\partial f_2}{\partial s} \tau_g - f_3 \tau_g^2 \\ \frac{\partial \tau_g}{\partial t} &= -\frac{\partial f_2}{\partial s} k_n + f_3 k_n \tau_g + \frac{\partial \psi}{\partial s} \end{aligned}$$

and

$$\psi k_n = \left(-f_1 k_n - \frac{\partial f_3}{\partial s} - f_2 \tau_g \right) \tau_g.$$

ii- If the curve α is a asymptotic line, we have

$$\begin{aligned} \frac{\partial k_g}{\partial t} &= \frac{\partial f_1}{\partial s} k_g + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s} \tau_g - f_3 \frac{\partial \tau_g}{\partial s} - \frac{\partial f_3}{\partial s} \tau_g - f_2 \tau_g^2 \\ \frac{\partial \tau_g}{\partial t} &= \frac{\partial \psi}{\partial s} \end{aligned}$$

and

$$\psi k_g = \left(-f_1 k_g - \frac{\partial f_2}{\partial s} + f_3 \tau_g \right) \tau_g.$$

iii- If the curve α is a curvature line, then we have

$$\begin{aligned} \frac{\partial k_g}{\partial t} &= \frac{\partial f_1}{\partial s} k_g + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} \\ \frac{\partial k_n}{\partial t} &= \frac{\partial f_1}{\partial s} k_n + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2}. \end{aligned}$$

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